Primal Hybrid Finite Element Methods
for 2nd Order Elliptic Equations

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Abstract. The paper is devoted to the construction of finite element methods for
2nd order elliptic equations based on a primal hybrid variational principle. Optimal
error bounds are proved. As a corollary, we obtain a general analysis of nonconform-
ing finite element methods.

1. Introduction. Let Ω be a bounded open subset of $\mathbf{R}^n$, with a Lipschitz continu-
ous boundary Γ. We consider the 2nd order elliptic model problem

(1.1)
$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \Gamma,$$

where f is a given function of the space $L^2(\Omega)$. The usual variational form of problem
(1.1) consists in finding $u \in H_0^1(\Omega)$ which minimizes the energy functional

(1.2)
$$J(v) = \frac{1}{2} \int_\Omega |\text{grad } v|^2 \, dx - \int_\Omega fv \, dx$$

over the space $H_0^1(\Omega)$.

Standard finite element methods for numerically solving problem (1.1) are based
on this variational principle: they consist in first constructing a finite-dimensional sub-
space V_h of the space $H_0^1(\Omega)$ made up with elementwise smooth functions which are
continuous along the interelement boundaries and then in minimizing the energy func-
tional $J(v)$ over the subspace V_h. Such *conforming* methods have been extensively
studied and convergence results are now classical (see for instance Ciarlet [4], Strang
and Fix [15]).

On the other hand, it has been noticed that one could weaken the requirement
of interelement continuity for the functions of the space V_h and still obtain a con-
vergent finite element method. One gets the so-called *nonconforming* methods in
which the space V_h is no longer contained in $H_0^1(\Omega)$. For an analysis of some non-
conforming methods for solving 2nd order elliptic problems, we refer to Crouzeix and
Raviart [6], Irons and Razzaque [8], Lesaint [9], Strang [14].

In this paper, we shall use a more general approach in order to construct finite
element approximations of problem (1.1). It is based on an extended variational
principle, known as the *primal hybrid principle,* in which the constraint of interelement
continuity has been removed at the expense of introducing a Lagrange multiplier.
This type of method has been first introduced by the engineers (cf. Pian [10], [11],

Pian and Tong [12]) and can be viewed as a generalization of nonconforming methods.

Similarly, dual hybrid methods can be derived by using the complementary energy principle (cf. again [10], [11], [12]). For the numerical analysis of these methods, we refer to Thomas [16], [17], [18]. Hybrid methods for solving 4th order elliptic problems have been also analyzed: see Brezzi [1], [2] and Brezzi and Marini [3]. Finally, we refer to Fraeijs de Veubeke [7] for a general discussion of finite element methods including hybrid methods.

An outline of the paper is as follows. In Section 2, we describe the primal hybrid variational principle associated with problem (1.1); in Section 3, we define the method of approximation. Examples of triangular and quadrilateral hybrid elements are derived in Sections 4 and 5, respectively. The error analysis is given in Section 6 and, in Section 7, we discuss briefly the use of numerical integration in order to derive nonconforming methods.

Some results of this paper have been announced in [13]. For the sake of conciseness, we have omitted some proofs and developments; they will be found in [18].

Throughout this paper, we shall make a constant use of the Sobolev spaces

$$H^m(\Omega) = \{v \in L^2(\Omega); \partial^\alpha v \in L^2(\Omega), |\alpha| \leqslant m\}$$

provided with the norm and seminorms

$$\|v\|_{m,\Omega} = \left(\sum_{|\alpha| \leqslant m} \int_\Omega |\partial^\alpha v|^2 \, dx \right)^{1/2}, \qquad |v|_{m,\Omega} = \left(\sum_{|\alpha| = m} \int_\Omega |\partial^\alpha v|^2 \, dx \right)^{1/2}.$$

Given a vector-valued function $\mathbf{q} = (q_1, \ldots, q_n) \in (H^m(\Omega))^n$, we shall set:

$$\|\mathbf{q}\|_{m,\Omega} = \left(\sum_{i=1}^n \|q_i\|_{m,\Omega}^2 \right)^{1/2}, \qquad |\mathbf{q}|_{m,\Omega} = \left(\sum_{i=1}^n |q_i|_{m,\Omega}^2 \right)^{1/2}.$$

We shall denote by $H^{1/2}(\Gamma)$ the space of the traces $v_{|\Gamma}$ over Γ of the functions $v \in H^1(\Omega)$ and we shall define as usual

$$H_0^1(\Omega) = \{v \in H^1(\Omega); v_{|\Gamma} = 0\}.$$

2. **The Primal Hybrid Variational Formulation.** Let $\overline{\Omega} = \bigcup_{r=1}^R \overline{\Omega}_r$ be a decomposition of the domain $\overline{\Omega}$ into subdomains $\overline{\Omega}_r$ such that:

(i) Ω_r is an open subset of Ω with a Lipschitz continuous boundary $\partial\Omega_r$, $1 \leqslant r \leqslant R$;

(ii) $\Omega_r \cap \Omega_s = \varnothing$ for $r \neq s$.

Clearly a function $v \in L^2(\Omega)$ belongs to the space $H_0^1(\Omega)$ iff

(a)   the restriction v_r of v to the set Ω_r belongs to the space $H^1(\Omega_r)$,

(b)   the traces of the functions v_r and v_s coincide on $\partial\Omega_r \cap \partial\Omega_s$;

(c)   the trace of the function v_r vanishes on $\partial\Omega_r \cap \Gamma$, $1 \leqslant r \leqslant R$.

We want to relax the conditions (b) and (c). Hence, we introduce the space

(2.1) $$X = \{v \in L^2(\Omega); v_r \in H^1(\Omega_r), 1 \leqslant r \leqslant R\} \approx \prod_{r=1}^R H^1(\Omega_r)$$

provided with the norm

(2.2)
$$\|v\|_X = \left(\sum_{r=1}^{R} \|v_r\|_{1,\Omega}^2 \right)^{1/2}.$$

In order to characterize $H_0^1(\Omega)$ as a subspace of the space X, we first introduce the space

(2.3)
$$\mathbf{H}(\text{div}; \Omega) = \{\mathbf{q} \in (L^2(\Omega))^n ; \text{div } \mathbf{q} \in L^2(\Omega)\}$$

normed by

(2.4)
$$\|\mathbf{q}\|_{\mathbf{H}(\text{div};\Omega)} = (\|\mathbf{q}\|_{0,\Omega}^2 + \|\text{div } \mathbf{q}\|_{0,\Omega}^2)^{1/2}.$$

Given a vector-valued function $\mathbf{q} \in \mathbf{H}(\text{div}; \Omega)$, we may define its normal component $\mathbf{q} \cdot \nu \in H^{-1/2}(\Gamma)$ where $H^{-1/2}(\Gamma)$ is the dual space of $H^{1/2}(\Gamma)$ and ν is the unit outward normal along Γ. Moreover, we have Green's formula

(2.5)
$$\forall v \in H^1(\Omega), \quad \int_{\Omega} \{\text{grad } v \cdot \mathbf{q} + v \text{ div } \mathbf{q}\} \, dx = \int_{\Gamma} v\mathbf{q} \cdot \nu \, d\gamma,$$

where the integral $\int_{\Gamma}$ represents the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

Next, we define the space

(2.6)
$$M = \left\{ \mu \in \prod_{r=1}^{R} H^{-1/2}(\partial\Omega_r); \text{ there exists a function } \mathbf{q} \in \mathbf{H}(\text{div}; \Omega) \right.$$
$$\left. \text{such that } \mathbf{q} \cdot \nu_r = \mu \text{ on } \partial\Omega_r, 1 \leqslant r \leqslant R \right\},$$

where ν_r is the unit outward normal along $\partial\Omega_r$. We may provide the space M with the norm

(2.7)
$$\|\mu\|_M = \inf_{\mathbf{q}\in\mathbf{H}(\text{div};\Omega); \, \mathbf{q}\cdot\nu_r=\mu \text{ on } \partial\Omega_r, 1\leqslant r\leqslant R} \|\mathbf{q}\|_{\mathbf{H}(\text{div};\Omega)}.$$

Then we have

LEMMA 1. *A continuous linear functional L on the space X vanishes on $H_0^1(\Omega)$ iff there exists a unique element $\mu \in M$ such that*

(2.8)
$$\forall v \in X, \quad L(v) = \sum_{r=1}^{R} \int_{\partial\Omega_r} \mu v \, d\gamma.$$

Proof. By the Hahn-Banach theorem, any continuous linear functional on $H^1(\Omega_r)$ is of the form

$$v \longrightarrow \int_{\Omega_r} \left\{ \sum_{i=1}^{n} q_i \frac{\partial v}{\partial x_i} + q_0 v \right\} dx, \quad q_i \in L^2(\Omega_r), 0 \leqslant i \leqslant n.$$

Hence, given a continuous linear functional L on X, there exist $n + 1$ functions $q_i \in L^2(\Omega)$, $0 \leqslant i \leqslant n$, such that

$$\forall v \in X, \quad L(v) = \sum_{r=1}^{R} \int_{\Omega_r} \left\{ \sum_{i=1}^{n} q_i \frac{\partial v}{\partial x_i} + q_0 v \right\} dx = 0.$$

Assuming that L vanishes on $H_0^1(\Omega)$, we get

$$\forall v \in H_0^1(\Omega), \quad L(v) = \int_\Omega \left\{ \sum_{i=1}^n q_i \frac{\partial v}{\partial x_i} + q_0 v \right\} dx = 0$$

so that $q_0 = \Sigma_{i=1}^n \, \partial q_i / \partial x_i$, in the sense of distributions in Ω. Setting $\mathbf{q} = (q_1, \ldots, q_n)$, we obtain div $\mathbf{q} = q_0 \in L^2(\Omega)$ so that $\mathbf{q} \in \mathbf{H}(\text{div}; \Omega)$ and

$$(2.9) \qquad \forall v \in X, \quad L(v) = \sum_{r=1}^R \int_{\Omega_r} \{ \mathbf{grad}\, v \cdot \mathbf{q} + v \, \text{div}\, \mathbf{q} \} \, dx.$$

Conversely, any linear functional of the form (2.9) is continuous on X and vanishes on $H_0^1(\Omega)$.

Now, using the Green's formula (2.5) in each Ω_r, we get from (2.9)

$$(2.10) \qquad \forall v \in X, \quad L(v) = \sum_{r=1}^R \int_{\partial \Omega_r} v \mathbf{q} \cdot \nu_r \, d\gamma.$$

Clearly, in (2.10), the function $\mathbf{q}$ is not uniquely determined but the corresponding element $\mu \in M$ is unique. In fact, assume that

$$\forall v \in X, \quad \sum_{r=1}^R \int_{\partial \Omega_r} \mu v \, d\gamma = 0.$$

Then, we get, for all $r = 1, \ldots, R$,

$$\forall v \in H^1(\Omega_r), \quad \int_{\partial \Omega_r} \mu v \, d\gamma = 0,$$

which implies $\mu = 0$ on $\partial \Omega_r$ by the surjectivity of the trace operator $v \longrightarrow v_{|\partial \Omega_r}$ from $H^1(\Omega_r)$ onto $H^{1/2}(\partial \Omega_r)$. The proof of the lemma is now complete. $\square$

Consider the continuous bilinear form on $X \times M$

$$(2.11) \qquad b(v, \mu) = - \sum_{r=1}^R \int_{\partial \Omega_r} \mu v \, d\gamma.$$

Then, as a consequence of Lemma 1, we get the following characterization of the space $H_0^1(\Omega)$

$$(2.12) \qquad H_0^1(\Omega) = \{ v \in X; \forall \mu \in M, \, b(v, \mu) = 0 \}.$$

We are now able to introduce the primal hybrid formulation of problem (1.1). Define the continuous bilinear form on $X \times X$

$$(2.13) \qquad a(u, v) = \sum_{r=1}^R \int_{\Omega_r} \mathbf{grad}\, u \cdot \mathbf{grad}\, v \, dx.$$

We want to find a pair $(u, \lambda) \in X \times M$ such that

$$(2.14) \qquad \forall v \in X, \quad a(u, v) + b(v, \lambda) = \int_\Omega fv \, dx,$$

$$(2.15) \qquad \forall \mu \in M, \quad b(u, \mu) = 0.$$

THEOREM 1. *Problem* (2.14), (2.15) *has a unique solution* $(u, \lambda) \in X \times M$. *Moreover* $u \in H_0^1(\Omega)$ *is the solution of problem* (1.1) *and we have*

(2.16) $\lambda = \partial u / \partial v_r \quad on \; \partial \Omega_r, \; 1 \leqslant r \leqslant R.$

Proof. Let $(u, \lambda) \in X \times M$ be a solution of (2.14), (2.15). Then, by (2.12) we have $u \in H_0^1(\Omega)$. Choosing $v \in H_0^1(\Omega)$ in (2.14) gives

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega \mathbf{grad} \; u \cdot \mathbf{grad} \; v \; dx = \int f v \; dx$$

so that u is the solution of problem (1.1). Conversely, let $u \in H_0^1(\Omega)$ be the solution of (1.1) and consider the continuous linear functional on X

$$v \longrightarrow \int_\Omega f v \; dx - a(u, v);$$

it vanishes on $H_0^1(\Omega)$ so that, by Lemma 1, there exists a unique $\lambda \in M$ such that

$$\forall v \in X, \quad b(v, \lambda) = \int_\Omega f v \; dx - a(u, v).$$

Hence, the pair (u, λ) is the solution of (2.14), (2.15).

Now, since $f = -\Delta u$, we obtain by using the Green's formula (2.5) in each Ω_r with $\mathbf{q} = \mathbf{grad} \; u$

$$\forall v \in X, \quad b(v, \lambda) = - \int_\Omega \Delta u v \; dx - a(u, v) = - \sum_{r=1}^R \int_{\partial \Omega_r} \frac{\partial u}{\partial v_r} v \; d\gamma$$

so that (2.16) holds. □

Remark 1. Define the continuous quadratic functional on $X \times M$

$$L(v, \mu) = J(v) + b(v, \mu).$$

Then, one can easily check that the solution $(u, \lambda) \in X \times M$ of problem (2.14), (2.15) may be characterized as the unique saddle-point of the functional $L(v, \mu)$ over $X \times M$, i.e.,

$$L(u, \lambda) = \underset{v \in X}{\text{Min}} \; \underset{\mu \in M}{\text{Max}} \; L(v, \mu) = \underset{\mu \in M}{\text{Max}} \; \underset{v \in X}{\text{Min}} \; L(v, \mu).$$

Hence, λ is the Lagrange multiplier associated with the constraint $u \in H_0^1(\Omega)$. □

3. A Hybrid Finite Element Method. Let us now introduce a method of approximation of problem (1.1) based on the primal hybrid variational formulation (2.14), (2.15). Given two finite-dimensional spaces X_h and M_h which satisfy the inclusions

(3.1) $X_h \subset X, \quad M_h \subset M,$

we define *problem* (Q_h): Find a pair $(u_h, \lambda_h) \in X_h \times M_h$ such that

(3.2) $\forall v_h \in X_h, \quad a(u_h, v_h) + b(v_h, \lambda_h) = \int_\Omega f v_h \; dx,$

(3.3) $\forall \mu_h \in M_h, \quad b(u_h, \mu_h) = 0.$

Before solving problem (Q_h), we introduce the space

(3.4) $V_h = \{v_h \in X_h; \forall \mu_h \in M_h, \, b(v_h, \mu_h) = 0\}.$

Comparing with (2.12), the space V_h appears to be an approximation of the space $H_0^1(\Omega)$. Notice however that V_h is not in general a subspace of $H_0^1(\Omega)$. Then, we define *problem* (P_h): Find $u_h \in V_h$ such that

(3.5) $\forall v_h \in V_h, \quad a(u_h, v_h) = \int_\Omega f v_h \, dx.$

Since in general $V_h \not\subset H_0^1(\Omega)$, problem (P_h) is a *nonconforming* method for numerically solving problem (1.1).

   Note that if $(u_h, \lambda_h) \in X_h \times M_h$ is a solution of problem (Q_h), $u_h \in V_h$ is a solution of problem (P_h). Moreover, we have the following result.

   THEOREM 2. *Assume that*

(3.6) $v_h \rightarrow \|v_h\|_h = a(v_h, v_h)^{1/2}$ *is a norm over* V_h.

Then:

   (i)   *problem* (P_h) *has a unique solution* $u_h \in V_h$;
   (ii) *problem* (Q_h) *has a unique solution* $(u_h, \lambda_h) \in X_h \times M_h$ *iff the following compatibility condition holds*

(3.7) $\{\mu_h \in M_h; \forall v_h \in X_h, \, b(v_h, \mu_h) = 0\} = \{0\}.$

   Proof. The existence and uniqueness of the solution $u_h \in V_h$ of problem (P_h) follows from the assumption (3.6) and the Lax-Milgram lemma. On the other hand, since problem (Q_h) is equivalent to a $N \times N$ linear system with $N = \dim X_h + \dim M_h$, the existence of the solution of problem (Q_h) follows from the uniqueness. Thus, assume $f = 0$. Necessarily $u_h = 0$ so that λ_h is characterized by the condition

$$\forall v_h \in X_h, \quad b(v_h, \lambda_h) = 0.$$

Therefore $\lambda_h = 0$ iff condition (3.7) holds. □

   In the sequel, we shall assume that Ω is a bounded and *polyhedral* subset of $\mathbf{R}^n$. Let T_h be a triangulation of the set $\overline{\Omega}$ with polyhedra K whose diameters are $\leqslant h$. For any $K \in T_h$, we denote by ∂K the boundary of K and ν_K the unit outward normal along ∂K. We now use the decomposition

$$\overline{\Omega} = \bigcup_{K \in T_h} K$$

of the domain $\overline{\Omega}$ for defining a hybrid finite element method. The first step consists in constructng a finite-dimensional subspace X_h of the space

$$X = X(T_h) = \{v \in L^2(\Omega); \forall K \in T_h, v_{|K} \in H^1(K)\}.$$

Let $\hat{K}$ be a *reference* polyhedron (for instance the unit right n-simplex or the unit hypercube). We assume for simplicity that each polyhedron $K \in T_h$ is the image of $\hat{K}$ through an affine invertible mapping F_K. We now introduce a finite-dimensional subspace $\hat{P}$ of the space $H^1(\hat{K})$ which satisfies the inclusion

(3.8) $P_k \subset \hat{P}$ for some integer $k \geqslant 1,$

where P_k denotes the space of all polynomials of degree $\leqslant k$ in the n variables x_1, $\ldots , x_n$.

Moreover, let us denote by $\hat{S}_k$ the space of all functions defined over $\partial \hat{K}$ whose restrictions to any $(n - 1)$-dimensional face $\hat{K}'$ of $\partial \hat{K}$ are polynomials of degree $\leqslant k$. Let us denote by $\hat{T}_k$ the space of all functions of S_k which are continuous over $\partial \hat{K}$. We assume, in addition, that the space $\hat{P}_{|\partial \hat{K}}$ of the restrictions to the set $\partial \hat{K}$ of all functions of $\hat{P}$ satisfies the inclusion

$$(3.9) \qquad \qquad \hat{T}_k \subset \hat{P}_{|\partial \hat{K}}.$$

Then, we define

$$(3.10) \qquad \qquad P_K = \{v \in H^1(K); v = \hat{v}_0 F_K^{-1}, \hat{v} \in \hat{P}\}$$

and

$$(3.11) \qquad \qquad X_h = \{v \in L^2(\Omega); \forall K \in T_h, v_{|K} \in P_K\}.$$

Note that the functions of X_h do not satisfy any continuity constraint at the interelement boundaries.

The second step consists in constructing a finite-dimensional subspace M_h of the space

$$M = M(T_h) = \left\{ \mu \in \prod_{K \in T_h} H^{-1/2}(\partial K); \text{ there exists a function } \mathbf{q} \in \mathbf{H}(\mathrm{div}; K) \right.$$
$$\left. \text{such that } \mathbf{q} \cdot \nu_K = \mu \text{ on } \partial K, K \in T_h \right\}.$$

We introduce a finite-dimensional subspace $\hat{S}$ of the space $L^2(\partial \hat{K})$ which satisfies the following properties:

$$(3.12) \qquad \qquad \hat{S}_m \subset \hat{S} \quad \text{for some integer } m \geqslant 0,$$

$$(3.13) \qquad \qquad \forall \hat{\alpha} \in \hat{S}_0, \quad \hat{\alpha}\hat{S} = \{\hat{\alpha}\hat{\mu}; \hat{\mu} \in \hat{S}\} \subset \hat{S}.$$

Now, for any $K \in T_h$, we set

$$(3.14) \qquad \qquad S_{\partial K} = \{\mu \in L^2(\partial K); \mu = \hat{\mu}_0 F_K^{-1}, \hat{\mu} \in \hat{S}\}.$$

Then, we define

$$M_h = \left\{ \mu \in \prod_{K \in T_h} S_{\partial K}; \mu_{|\partial K_1} + \mu_{|\partial K_2} = 0 \text{ on } K_1 \cap K_2 \text{ for} \right.$$
$$(3.15)$$
$$\left. \text{every pair of adjacent elements } K_1, K_2 \in T_h \right\}.$$

One easily checks that M_h is indeed a subspace of M.

Next, we want to give some simple sufficient conditions for hypotheses (3.6) and (3.7) to hold. We begin with

LEMMA 2. *Assume that the inclusion* (3.12) *holds. Then, condition* (3.6) *is satisfied.*

Proof. Let v_h be a function of V_h such that

$$\|v_h\|_h^2 = a(v_h, v_h) = \sum_{K \in T_h} \int_K |\text{grad } v_h|^2 \, dx = 0.$$

Clearly, the function v_h is constant in each element $K \in T_h$. Now, let K' be a $(n-1)$-dimensional common face of two adjacent elements $K_1, K_2 \in T_h$. Define $\mu_h \in \Pi_{K \in T_h} L^2(\partial K)$ by

$$\mu_{h | \partial K_1} = \begin{cases} 1 & \text{on } K', \\ 0 & \text{on } \partial K_1 \backslash K', \end{cases} \qquad \mu_{h | \partial K_2} = \begin{cases} -1 & \text{on } K', \\ 0 & \text{on } \partial K_2 \backslash K', \end{cases}$$

$$\mu_{h | \partial K} = 0 \quad \text{for } K \neq K_1, K_2.$$

Since $\hat{S}_0 \subset \hat{S}$, the function μ_h belongs to the space M_h. Hence, denoting by c_i the constant value of v_h in K_i, $i = 1, 2$, and using the definition (3.4) of the space V_h, we get

$$0 = b(v_h, \mu_h) = (c_1 - c_2) \int_{K'} d\gamma$$

so that $c_1 = c_2$. Therefore, the function v_h is constant in Ω.

Finally, let K' be a $(n-1)$-dimensional face contained in Γ. Define the function $\mu_h \in M_h$ by

$$\mu_{h | K'} = 1, \quad \mu_h = 0 \quad \text{elsewhere.}$$

We obtain

$$0 = b(v_h, \mu_h) = \int_{K'} v_h \, d\gamma$$

so that $v_h = 0$ in Ω. This proves that $v_h \longrightarrow \|v_h\|_h$ is a norm over the space V_h. $\square$

Since condition (3.6) is always satisfied, problem $(\mathbf{P}_h)$ has a unique solution $u_h \in V_h$. On the contrary, condition (3.7) which ensures the existence and uniqueness of the Lagrange multiplier $\lambda_h \in M_h$ is satisfied only for compatible choices of the spaces $\hat{P}$ and $\hat{S}$. In this respect, we have

LEMMA 3. *Assume that*

$$(3.16) \qquad \left\{ \mu \in \hat{S}; \ \forall v \in \hat{P}, \ \int_{\partial \hat{K}} \mu v \, d\gamma = 0 \right\} = \{0\}.$$

Then, condition (3.7) holds.

Proof. Using (3.16) and the definitions (3.10) and (3.14) of the spaces P_K and $S_{\partial K}$ respectively, we get for all $K \in T_h$

$$\left\{ \mu \in S_{\partial K}; \ \forall v \in P_K, \ \int_{\partial K} \mu v \, d\gamma = 0 \right\} = \{0\}.$$

This in turn implies (3.7). $\square$

Remark 2. In fact, one can prove that, at least for some particular triangulation T_h of the domain $\overline{\Omega}$, condition (3.16) is as well necessary for hypothesis (3.7) to hold. $\square$

The next two sections of this paper will now be devoted to the explicit construction of triangular and quadrilateral finite elements in R^2 which satisfy the compatability condition (3.16).

4. Examples of Hybrid Triangular Elements. Assume now $n = 2$. We first consider the case of triangular elements: here $\hat{K}$ is a triangle with vertices a_i, $1 \leqslant i \leqslant 3$; we denote by $\lambda_i = \lambda_i(x)$, $1 \leqslant i \leqslant 3$, the barycentric coordinates of a point $x \in \mathbf{R}^2$ with respect to the vertices a_i of $\hat{K}$.

LEMMA 4. *Assume that $\hat{K}$ is a triangle. Let $k \geqslant 1$ and $m \geqslant 0$ be two integers. Then conditions*

$$(4.1) \qquad \begin{cases} \mu \in \hat{S}_m, \\[2mm] \forall v \in \hat{T}_k, \ \int_{\partial \hat{K}} \mu v \, d\gamma = 0, \end{cases}$$

imply $\mu = 0$ iff

$$(4.2) \qquad k \geqslant \begin{cases} m + 1 & \text{when m is even,} \\[2mm] m + 2 & \text{when m is odd.} \end{cases}$$

Proof. Since $\hat{K}$ is a triangle, we have $\hat{T}_k = P_{k|\partial \hat{K}}$ and $\dim(\hat{S}_m) = 3(m + 1)$, $\dim(\hat{T}_k) = 3k$.

For $k \leqslant m$, we get $\dim(\hat{S}_m) > \dim(\hat{T}_k)$ so that conditions (4.1) cannot imply $\mu = 0$.

Thus, let us assume $k \geqslant m + 1$ and let μ satisfy conditions (4.1). Then, for all $v \in \hat{T}_k$, we get $\mu v \in \hat{S}_{2k-1}$ so that the integral $\int_{\partial \hat{K}} \mu v \, d\gamma$ can be computed exactly in terms of the values of the function μv at the $(k + 1)$ *Gauss-Lobatto quadrature points* of each side of $\partial \hat{K}$. Denote by $\{a_1, a_4, \ldots, a_{k+2}, a_2\}$ (resp. $\{a_2, a_{k+3}, \ldots, a_{2k+1}, a_3\}$, $\{a_3, a_{2k+2}, \ldots, a_{3k}, a_1\}$ the set of $(k + 1)$ Gauss-Lobatto points of the side $[a_1, a_2]$ (resp. $[a_2, a_3]$, $[a_3, a_1]$). Clearly, for each $i = 1, \ldots, 3k$, there exists a unique function $v_i \in \hat{T}_k$ such that

$$v_i(a_j) = \delta_{ij}, \qquad 1 \leqslant j \leqslant 3k.$$

Since conditions (4.1) are invariant by an affine invertible mapping, we may assume that the triangle $\hat{K}$ is *equilateral*. Then, replacing v by v_i, $1 \leqslant i \leqslant 3k$, in (4.1) gives

$$(4.3) \qquad \mu(a_i), \qquad 4 \leqslant i \leqslant 3k,$$

and

$$(4.4) \qquad \begin{cases} \mu_{12}(a_1) + \mu_{13}(a_1) = 0, \\[2mm] \mu_{23}(a_2) + \mu_{21}(a_2) = 0, \\[2mm] \mu_{31}(a_3) + \mu_{32}(a_2) = 0, \end{cases}$$

where $\mu_{ij} = \mu_{ji}$ is the restriction of μ to the side $[a_i, a_j]$.

For $k \geqslant m + 2$, conditions (4.3) imply $\mu = 0$. Hence, it remains only to consider the case $k = m + 1$. Let

$$0 = \xi_0 < \xi_1 < \cdots < \xi_{k-1} < \xi_k = 1$$

be the $(k + 1)$ Gauss-Lobatto abscissae for $[0, 1]$; we introduce the homogeneous polynomial of degree $k - 1$ in the variables ξ and η

$$(4.5) \qquad p_{k-1}(\xi, \eta) = \prod_{i=1}^{k-1} (\eta_i \xi - \xi_i \eta),$$

where $\eta_i = 1 - \xi_i$, $1 \leqslant i \leqslant k - 1$. Since $\eta_i = \xi_{k-i}$, $1 \leqslant i \leqslant k - 1$, we get

$$p_{k-1}(\xi, \eta) = (-1)^{k-1} p_{k-1}(\eta, \xi).$$

Thus, for $k = m + 1$, conditions (4.3) exactly mean

$$(4.6) \qquad \mu_{ij} = c_{ij} p_{k-1}(\lambda_i, \lambda_j) \quad \text{on } [a_i, a_j], \qquad c_{ij} = (-1)^{k-1} c_{ji}.$$

Using (4.6), conditions (4.4) become

$$(4.7) \qquad \begin{cases} c_{12} + (-1)^{k-1} c_{31} = 0, \\ c_{23} + (-1)^{k-1} c_{12} = 0, \\ c_{31} + (-1)^{k-1} c_{23} = 0. \end{cases}$$

When $m = k - 1$ is even, the linear system (4.7) has the unique solution $c_{12} = c_{23} = c_{31} = 0$, so that $\mu = 0$. When $m = k - 1$ is odd, (4.7) has nontrivial solutions so that conditions (4.1) do not imply $\mu = 0$. $\square$

  Remark 3. When $m = k - 1$ is odd, we have proved that the space of functions μ which satisfy conditions (4.1) is one-dimensional. When $\hat{K}$ is equilateral, these functions μ are the functions of the form

$$(4.8) \qquad \begin{cases} \mu_{12} = c p_{k-1}(\lambda_1, \lambda_2), \\ \mu_{23} = c p_{k-1}(\lambda_2, \lambda_3), \qquad c \in \mathbf{R}. \; \square \\ \mu_{31} = c p_{k-1}(\lambda_3, \lambda_1), \end{cases}$$

We are now able to introduce

  Example 1. Let $k \geqslant 1$ be an *odd* integer. Given a reference triangle $\hat{K}$, we choose:

$$\hat{P} = P_k, \qquad \hat{S} = \hat{S}_{k-1} \qquad (m = k - 1).$$

Then, by Lemmas 2, 3 and 4, the corresponding problem (Q_h) has a unique solution (u_h, λ_h).

  Let us characterize the associated space V_h defined by (3.4). Clearly a function $v_h \in X_h$ belongs to the space V_h iff:

  (i) for any pair (K_1, K_2) of adjacent triangles of T_h, we have

$$\forall \mu \in P_{k-1}, \quad \int_{K'} \mu(v_{h,1} - v_{h,2}) \, d\gamma = 0,$$

where $K' = K_1 \cap K_2$ and $v_{h,i}$ is the restriction of v_h to K_i, $i = 1, 2$;

  (ii) for any side K' of T_h contained in Γ, we have

$$\forall \mu \in P_{k-1}, \quad \int_{K'} \mu v_h \, d\gamma = 0.$$

Now, in each above integral, $\mu v_{h|K'}$ belongs to the space $P_{2k-1|K'}$, so that $\int_{K'} \mu v_h \, d\gamma$ can be computed exactly in terms of the values of μv_h at the k *Gauss-Legendre quadrature points* of the side K'. Hence a function $v_h \in X_h$ belongs to V_h iff:

(4.9) v_h is continuous at these Gauss-Legendre points contained in Ω;

(4.10) v_h vanishes at these Gauss-Legendre points located on Γ.

Denote by $\{b_i\}_{i=1}^{k}$ (resp. $\{b_i\}_{i=k+1}^{2k}$, $\{b_i\}_{i=2k+1}^{3k}$) the set of the k Gauss-Legendre points of the side $[a_1, a_2]$ (resp. $[a_2, a_3]$, $[a_3, a_1]$).

LEMMA 5. *Assume that k is an odd integer. Then $\{b_i\}_{i=1}^{3k}$ is a $\hat{T}_k$-unisolvent set. Moreover, for $k \geqslant 3$ and for any P_{k-3}-unisolvent set $\{b_i\}_{i=3k+1}^{N(k)}$ of points of $\overset{\circ}{\hat{K}}$ ($N(k) = (k+1)(k+2)/2$), the set*

$$\hat{\Sigma} = \{b_i\}_{i=1}^{N(k)} \text{ is } P_k\text{-unisolvent.}*$$

Proof. Let us first show that $\{b_i\}_{i=1}^{3k}$ is a $\hat{T}_k$-unisolvent set. Since $\dim(\hat{T}_k) = 3k$, we have only to show that a function $v \in \hat{T}_k$ which vanishes at the points b_i, $1 \leqslant i \leqslant 3k$, must vanish identically. In fact, since k is odd, we get for such a function v

$$v(a_1) = -v(a_2) = v(a_3) = -v(a_1)$$

so that $v(a_i) = 0$, $1 \leqslant i \leqslant 3$. Hence, the restriction of v to any side $\hat{K}'$ of $\hat{K}$ is a polynomial of degree $\leqslant k$ which vanishes at $k + 2$ distinct points so that $v = 0$.

Next, assume $k \geqslant 3$; we get

$$N(k) = \dim(P_k) = 3k + \dim(P_{k-3}).$$

Let $\{b_i\}_{i=3k+1}^{N(k)}$ be a set of $\dim(P_{k-3})$ distinct points of $\overset{\circ}{\hat{K}}$. Then, given a function $v \in P_k$ which vanishes on $\hat{\Sigma} = \{b_i\}_{i=1}^{N(k)}$, we get by the first part of the proof $v_{|\partial\hat{K}} = 0$ so that

$$v = \lambda_1 \lambda_2 \lambda_3 w, \quad w \in P_{k-3}.$$

Therefore $v = 0$ iff $w = 0$ or, equivalently, the set $\hat{\Sigma}$ is P_k-unisolvent iff the set $\{b_i\}_{i=3k+1}^{N(k)}$ is P_{k-3}-unisolvent. □

Using the terminology of [4], we deduce from conditions (4.9), (4.10) and Lemma 5 that the space V_h is associated with a reference finite element $(\hat{K}, \hat{\Sigma}, P_k)$ where $\hat{\Sigma}$ is given (in a nonunique way for $k \geqslant 3$) by Lemma 5. Setting

$$\Sigma_K = F_K(\hat{\Sigma}), \quad K \in T_h, \quad \Sigma_h = \bigcup_{K \in T_h} \Sigma_K,$$

we see that the degrees of freedom of a function $v_h \in V_h$ are its values at the points of $\Sigma_h \cap \Omega$. Since $(\hat{K}, \hat{\Sigma}, P_k)$ is not a C^0-element, the space V_h is not contained in $H_0^1(\Omega)$ and problem (P_h) is indeed a nonconforming method for solving problem (1.1).

---

*Let us recall that a set $\Sigma = \{a_i\}_{i=1}^{N}$ is P-unisolvent if for any set of scalars α_i, $1 \leqslant i \leqslant N$, there exists a unique function $p \in P$ such that $p(a_i) = \alpha_i$, $1 \leqslant i \leqslant N$.

On the other hand, we may choose the degrees of freedom of a function $\mu_h \in M_h$ to be its values at the k Gauss-Legendre points of each side K' of T_h.

The first hybrid elements are described in Figure 1 where we have used the following conventions for the degrees of freedom:

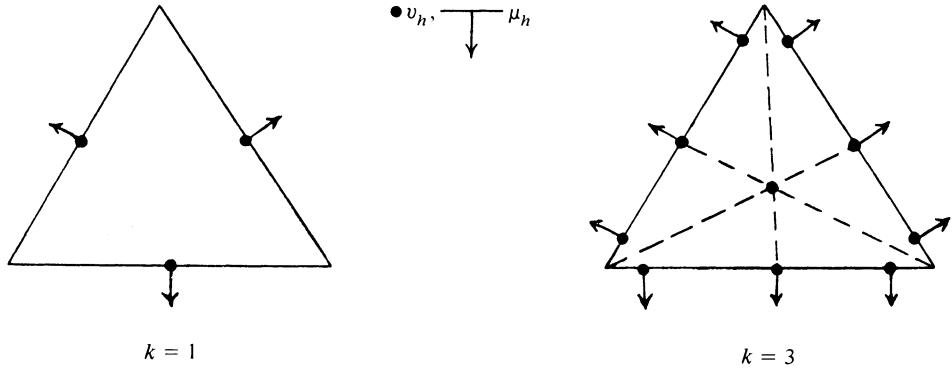

$$k = 1 \qquad\qquad\qquad k = 3$$

FIGURE 1

These hybrid elements can be considered as nonconforming elements using Loof connections (cf. Crouzeix and Raviart [6], Irons and Razzaque [8]).

Example 2. Let $k \geqslant 2$ be an *even* integer. By Lemma 4, the choice $\hat{P} = P_k$, $\hat{S} = \hat{S}_m$ is suitable only for $k \geqslant m + 2$. However, the next result will enable us to construct a hybrid method where

$$P_k \subset \hat{P} \subset P_{k+1}, \qquad \hat{S} = \hat{S}_{k-1}.$$

LEMMA 6. *Assume that k is an even integer. Define $\hat{P}$ to be the space of polynomials spanned by P_k and the function*

$$
\begin{gathered}
v_0 = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)\{(\lambda_1\lambda_2)^{(k-2)/2} \\
+ (\lambda_2\lambda_3)^{(k-2)/2} + (\lambda_3\lambda_1)^{(k-2)/2}\}.
\end{gathered}
$$

(4.11)

Then, the pair of spaces $(\hat{P}, \hat{S}_{k-1})$ satisfies condition (3.15).

Proof. Let μ be a function of $\hat{S}_{k-1}$ such that

$$\forall v \in \hat{P}, \quad \int_{\partial \hat{K}} \mu v \, d\gamma = 0.$$

Assume again for convenience that the triangle $\hat{K}$ is equilateral. Then, by Remark 3, μ is necessarily of the form (4.8). Hence, it is sufficient to prove that

$$\sum_{i=1}^{3} \int_{[a_i, a_{i+1}]} p_{k-1}(\lambda_i, \lambda_{i+1}) v_0 d\gamma \neq 0 \qquad (a_4 = a_1, \lambda_4 = \lambda_1).$$

Since these three integrals are equal, we have only to check that

$$\int_{[a_1, a_2]} p_{k-1}(\lambda_1, \lambda_2) v_0 \, d\gamma \neq 0.$$

By an obvious change of variable, the previous integral becomes

$$\int_0^1 q_{k-1}(\xi) r(\xi) \xi (1 - \xi) \, d\xi,$$

where

$$q_{k-1}(\xi) = p_{k-1}(\xi, 1 - \xi), \qquad r(\xi) = (1 - 2\xi)(\xi(1 - \xi))^{(k-2)/2}.$$

Since the roots of q_{k-1} are the Gauss-Lobatto abscissae $\xi_1, \ldots, \xi_{k-1}$, the polynomial q_{k-1} is orthogonal to all polynomials of degree $\leqslant k - 2$ with respect to the weight function $\xi(1 - \xi)$. Now, r is a polynomial of degree $k - 1$ so that

$$\int_0^1 q_{k-1}(\xi) r(\xi) \xi(1 - \xi) d\xi \neq 0.$$

Otherwise, q_{k-1} would be orthogonal to all polynomials of degree $\leqslant k - 1$ which is clearly impossible. □

Now, we use the pair of spaces $(\hat{P}, \hat{S}_{k-1})$ defined in Lemma 6 in order to construct the spaces X_h and M_h associated with the triangulation T_h. Again, by Lemmas 2, 3 and 6, problem (Q_h) has a unique solution (u_h, λ_h).

Since, for $v_h \in X_h, \mu_h \in M_h$, we have $\mu_h v_{h|K'} \in P_{2k|K'}$, the integrals $\int_{K'} \mu_h v_h \, d\gamma$ cannot be anymore computed exactly in terms of the values of $\mu_h v_h$ at the k Gauss-Legendre points of the side K'. As a consequence, the space V_h is not associated here with a (nonconforming) reference finite element $(\hat{K}, \hat{\Sigma}, \hat{P})$ so that the degrees of freedom of a function $v_h \in V_h$ cannot be determined in a simple way. However, we shall see in Section 7 how the use of numerical quadrature for evaluating the various integrals $\int_{K'} \mu_h v_h \, d\gamma$ will enable us to solve this difficulty.

5. Examples of Hybrid Quadrilateral Elements. We now consider the case of quadrilateral elements: Here $\hat{K}$ is the unit square $[0, 1]^2$ in the (ξ, η)-plane with vertices $a_1 = (0, 0), a_2 = (1, 0), a_3 = (1, 1), a_4 = (0, 1)$. For any integer $k \geqslant 1$, we define Q_k to be the space of all polynomials of the form

$$p(\xi, \eta) = \sum_{0 \leqslant i, j \leqslant k} c_{ij}\xi^i\eta^j.$$

LEMMA 7. *Assume that $\hat{K}$ is the unit square. Let $k \geqslant 1$ be an integer. Then the space of functions $\mu \in \hat{S}_{k-1}$ such that*

(5.1) $$\forall v \in \hat{T}_k, \quad \int_{\partial \hat{K}} \mu v \, d\gamma = 0$$

is one-dimensional.

The proof goes along the same lines of that of Lemma 4. More precisely, using the notations of Section 4, one can show that the functions $\mu \in \hat{S}_{k-1}$ which satisfy condition (5.1) are given by

(5.2) $$\begin{cases} \mu_{12} = c_{12}p_{k-1}(\xi, 1 - \xi), \\[6pt] \mu_{23} = c_{23}p_{k-1}(\eta, 1 - \eta), \\[6pt] \mu_{34} = c_{34}p_{k-1}(1 - \xi, \xi), \\[6pt] \mu_{41} = c_{41}p_{k-1}(1 - \eta, \eta), \end{cases}$$

with

$$(5.3) \quad \begin{cases} c_{12} = -c_{23} = c_{34} = -c_{41} = c & \text{when } k \text{ is odd,} \\ \\ c_{12} = c_{23} = c_{34} = c_{41} = c & \text{when } k \text{ is even.} \end{cases}$$

Here again, μ_{ij} denotes the restriction of μ to the side $[a_i, a_j]$ of $\hat{K}$.

Now, in order to construct a suitable hybrid element, we introduce a space $\hat{Q}$ of functions defined over $\hat{K}$ such that

$$(5.4) \qquad\qquad P_k \subset \hat{Q}, \qquad \hat{T}_k \subset \hat{Q}_{|\partial \hat{K}}.$$

Since $\hat{T}_k = Q_{k|\partial\hat{K}}$, we can choose $\hat{Q} = Q_k$. More generally, we can choose "serendipity" space $\hat{Q}$ with $P_k \subset \hat{Q} \subset Q_k$: for the derivation of the "serendipity" spaces, we refer to [4] and [20, Chapter 7].

LEMMA 8.   *Let $k \geq 1$ be an integer. Define $\hat{P}$ to be the space of polynomials spanned by $\hat{Q}$ and the function*

$$(5.5)\, v_0(\xi, \eta) = \begin{cases} [\xi(1 - \xi) - \eta(1 - \eta)][(\xi(1 - \xi))^{(k-1)/2} + (\eta(1 - \eta))^{(k-1)/2}] \\ \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{when } k \textit{ is odd,} \\ \\ [\xi(1 - \xi) - \eta(1 - \eta)]\,(2\xi - 1)(2\eta - 1) \\ \\ \quad \cdot [(\xi(1 - \xi))^{(k-2)/2} + (\eta(1 - \eta))^{(k-2)/2}] \quad \textit{when } k \textit{ is even.} \end{cases}$$

Assume that condition (5.4) holds. Then the pair of spaces $(\hat{P}, \hat{S}_{k+1})$ satisfies the compatibility condition (3.16).

The proof is based upon Lemma 7 and is very similar to that of Lemma 6.

Example 3. Let $k \geq 1$ be *any* integer. Assume that T_h is a triangulation of $\overline{\Omega}$ made up with parallelograms K. On the reference unit square $\hat{K}$, we choose a pair of spaces $(\hat{P}, \hat{S}_{k-1})$ defined in Lemma 8 in order to construct the spaces X_h and M_h. By Lemmas 2, 3 and 8, problem (Q_h) has a unique solution.

Note that, as in Example 2, the space V_h is not associated with a (nonconforming) reference finite element $(\hat{K}, \hat{\Sigma}, \hat{P})$ but see again Section 7 for the use of numerical quadrature.

6. Error Estimates. Let us go back to the general situation of Section 3. We want to derive bounds for the errors $u - u_h$ and $\lambda - \lambda_h$. We begin by defining more convenient norms over the spaces X and M. We first provide X with the norm

$$(6.1) \qquad\qquad |||v|||_X = \left(\sum_{K \in T_h} |||v|||_{1,K}^2 \right)^{1/2}$$

where for any $K \in T_h$

$$(6.2) \qquad\qquad |||v|||_{1,K} = (|v|_{1,K}^2 + h_K^{-2}\|v\|_{0,K}^2)^{1/2},$$

$$(6.3) \qquad\qquad h_K = \text{diameter of } K.$$

Next, we introduce the following norm over the space M

$$(6.4) \qquad |||\mu|||_M = \sup_{v \in X} \frac{b(v, \mu)}{|||v|||_X}.$$

Remark 4. One can easily check that the natural norm (2.7) over M can be equivalently defined by

$$(6.5) \qquad \|\mu\|_M = \sup_{v \in X} \frac{b(v, \mu)}{\|v\|_X}.$$

Therefore, the two norms (2.7) and (6.4) are equivalent, the latter being more appropriate for deriving practical error bounds. □

Let us now state the following result which can be viewed as a variant of a general theorem of Brezzi concerning the approximation of variational problems (cf. [1, Theorem 2.1]).

THEOREM 3. *Assume that the hypotheses* (3.1) *and* (3.6) *hold. Then, the solution* $u_h \in V_h$ *of problem* $(\mathbf{P}_h)$ *satisfies*

$$(6.6) \quad \|u - u_h\|_h = \left\{ \left(\inf_{v_h \in V_h} \|u - v_h\|_h \right)^2 + \left(\inf_{\mu_h \in M_h} \sup_{\mu_h \in M_h} \frac{b(v_h, \lambda - \mu_h)}{\|v_h\|_h} \right)^2 \right\}^{1/2}.$$

Assume, in addition, that there exists a constant $\alpha > 0$ *such that*

$$(6.7) \qquad \forall \mu_h \in M_h, \quad \sup_{v_h \in X_h} \frac{b(v_h, \mu_h)}{|||v_h|||_X} \geq \alpha |||\mu_h|||_M.$$

Then, problem $(\mathbf{Q}_h)$ *has a unique solution* (u_h, λ_h) *and we have*

$$(6.8) \qquad |||\lambda - \lambda_h|||_M \leq \frac{1}{\alpha} \|u - u_h\|_h + \left(1 + \frac{1}{\alpha} \right) \inf_{\mu_h \in M_h} |||\lambda - \mu_h|||_M.$$

Proof. Assume that hypotheses (3.1) and (3.6) hold so that, by Theorem 2, problem $(\mathbf{P}_h)$ has a unique solution $u_h \in V_h$. Define $\pi_h u \in V_h$ by

$$\forall v_h \in V_h, \quad a(u - \pi_h u, v_h) = 0.$$

Thus, we may write

$$\|u - u_h\|_h^2 = \|u - \pi_h u\|_h^2 + \|\pi_h u - u\|_h^2.$$

On the other hand, since $\pi_h u$ is the orthogonal projection of u upon V_h with respect to the inner product $a(\cdot, \cdot)$, we get

$$\|u - \pi_h u\|_h = \inf_{v_h \in V_h} \|u - v_h\|_h.$$

On the other hand, we have

$$\|\pi_h u - u_h\|_h = \sup_{v_h \in V_h} \frac{a(\pi_h u - u_h, v_h)}{\|v_h\|_h} = \sup_{v_h \in V_h} \frac{a(u - u_h, v_h)}{\|v_h\|_h}.$$

Using (2.14), (3.4) and (3.5), we obtain for all $v_h \in V_h$ and all $\mu_h \in M_h$

$$a(u - u_h, v_h) = b(v_h, \lambda - \mu_h)$$

so that

$$\|\pi_h u - u_h\|_h = \inf_{\mu_h \in M_h} \sup_{v_h \in V_h} \frac{b(v_h, \lambda - \mu_h)}{\|v_h\|_h}.$$

This proves (6.6).

Next, we notice that condition (3.7) exactly means that there exists a constant $\alpha > 0$ (which may depend on X_h and M_h) such that (6.7) holds. Hence, by Theorem 2 again, the hypothesis (6.7) implies the existence and uniqueness of the solution (u_h, λ_h) of problem (Q_h). Now, using (2.14) and (3.2), we get for all $v_h \in X_h$ and all $\mu_h \in M_h$

$$b(v_h, \lambda_h - \mu_h) = a(u - u_h, v_h) + b(v_h, \lambda - \mu_h).$$

Given $\mu_h \in M_h$, we can choose by hypothesis (6.7) a function $v_h \in X_h$ such that

$$b(v_h, \lambda_h - \mu_h) \geqslant \alpha \|\|v_h\|\|_X \|\|\lambda_h - \mu_h\|\|_M.$$

Since $\|v_h\|_h \leqslant \|\|v_h\|\|_X$, we obtain

$$\alpha \|\|\lambda_h - \mu_h\|\|_M \leqslant \|u - u_h\|_h + \|\|\lambda - \mu_h\|\|_M.$$

Thus, we get, for all $\mu_h \in M_h$,

$$\|\|\lambda - \lambda_h\|\|_M \leqslant \frac{1}{\alpha} \|u - u_h\|_h + \left(1 + \frac{1}{\alpha}\right) \|\|\lambda - \mu_h\|\|_M$$

and the desired inequality (6.8) holds.

To apply the above results, we consider a *regular family* (T_h) of triangulations of the domain $\overline{\Omega}$ in the sense of [5], in that there exists a constant $\sigma > 0$ independent of h such that

$$(6.9) \qquad\qquad\qquad \max_{K \in T_h} \frac{h_K}{\rho_K} \leqslant \sigma,$$

where

$$(6.10) \qquad\qquad \rho_K = \text{diameter of the inscribed sphere in } K.$$

Given a function $\varphi \in H^2(\Omega)$, we define $\psi \in M$ by

$$(6.11) \qquad\qquad\qquad \psi = \frac{\partial \varphi}{\partial \nu_K} \quad \text{on } \partial K,\ K \in T_h.$$

We first evaluate the expression

$$\inf_{\mu_h \in M_h} \sup_{v \in X} \frac{b(v, \psi - \mu_h)}{\|v\|_h}.$$

LEMMA 9. *Let there be given spaces M_h defined as in (3.15) which are associated with a regular family of triangulations (T_h). Assume that the inclusion (3.12)*

holds and that $\varphi \in H^{l+1}(\Omega)$ with $1 \leqslant l \leqslant m + 1$. Then there exists a constant $T > 0$ independent of h such that

$$(6.12) \qquad \inf_{\mu_h \in M_h} \sup_{v \in X} \frac{b(v, \psi - \mu_h)}{\|v\|_h} \leqslant Th^l |\varphi|_{l+1,\Omega}.$$

Proof. Let us recall the following result [6, Lemma 3]: there exists a constant $c > 0$ such that, for any $K \in T_h$ and for any $(n-1)$-dimensional face K', we have

$$\forall \chi \in H^l(K), 1 \leqslant l \leqslant m + 1, \forall v \in H^l(K), \quad \int_{K'} (\chi - \pi_{K'}^m \chi) v \, d\gamma \leqslant c \frac{h_K^{l+1}}{\rho_K} |\chi|_{1,K} |v|_{1,K}$$

where $\pi_{K'}^m$ denotes the orthogonal projector in $L^2(K')$ upon $P_{m|K'}$.

Now, given $\varphi \in H^{l+1}(\Omega)$, we define $\mu_h \in M_h$ in the following way: for any $K \in T_h$ and any $(n-1)$-dimensional face K' of K, we set

$$\mu_h = \pi_{K'}^m \frac{\partial \varphi}{\partial \nu_K} = \pi_{K'}^m \psi \quad \text{on } K'.$$

By the previous inequality, we obtain for all $v \in H^1(K)$

$$\int_{K'} (\psi - \mu_h) v \, d\gamma = \int_{K'} \left(\frac{\partial \varphi}{\partial \nu_K} - \pi_{K'}^m \frac{\partial \varphi}{\partial \nu_K} \right) v \, d\gamma \leqslant c \frac{h_K^{l+1}}{\rho_K} |\varphi|_{l+1,K} |v|_{1,K}.$$

Hence, using (6.9), we get, for all $v \in X$,

$$b(v, \psi - \mu_h) = - \sum_{K \in T_h} \int_{\partial K} (\psi - \mu_h) v \, d\gamma \leqslant Th^l |\varphi|_{l+1,\Omega} \|v\|_h$$

and the conclusion follows immediately. $\square$

We now give an estimate of the error $\|u - u_h\|_h$.

THEOREM 4. *Let there be given spaces X_h and M_h defined as in (3.11) and (3.15), which are associated with a regular family of triangulations. Assume that the inclusions (3.8), (3.9), (3.12) hold and that $u \in H^{l+1}(\Omega) \cap H_0^1(\Omega)$ with $l + 1 - n/2 > 0$ and $1 \leqslant l \leqslant \min(k, m + 1)$. Then there exists a constant $T > 0$ independent of h such that*

$$(6.13) \qquad \|u - u_h\|_h \leqslant Th^l |u|_{l+1,\Omega}.$$

Proof. Define the subspace W_h of the space $H_0^1(\Omega)$ by

$$W_h = \{w_h \in C^0(\overline{\Omega}); \forall K \in T_h, w_{h|K} \in P_K, w_{h|\Gamma} = 0\}.$$

Since $M_h \subset M$, we have $W_h \subset V_h$. Now, in view of the inclusions (3.8), (3.9) and since (T_h) is a regular family, it is well known (see [5, Theorem 5], [4], for example) that, for $u \in H^{l+1}(\Omega)_n \cap H_0^l(\Omega)$ with $l + 1 - n/2 > 0$ and $1 \leqslant l \leqslant k$, we have

$$\inf_{w_h \in W_h} |u - w_h|_{1,\Omega} \leqslant c_1 h^l |u|_{l+1,\Omega},$$

where the constant c_1 is independent of h. Hence

$$(6.14) \qquad \inf_{v_h \in V_h} \|u - v_h\|_h \leqslant \inf_{w_h \in W_h} |u - v_h|_{1,\Omega} \leqslant c_1 h^l |u|_{l+1,\Omega}.$$

Using next Lemma 9 with $\varphi = u$, $\psi = \lambda$ and since $u \in H^{l+1}(\Omega)$ with $1 \leqslant l \leqslant m + 1$, we obtain for some constant c_2 independent of h

$$(6.15) \qquad \inf_{\mu_h \in M_h} \sup_{v_h \in V_h} \frac{b(v_h, \lambda - \mu_h)}{\|v_h\|_h} \leqslant c_2 h^l |u|_{l+1,\Omega}.$$

Then inequality (6.13) follows from inequalities (6.6), (6.14) and (6.15). □

   Remark 5. Assume in addition that the polyhedral domain $\bar{\Omega}$ is *convex*. Then, as in [6, Theorem 4], an extension of the classical duality technique of Aubin-Nitsche yields the following L^2-estimate

$$(6.16) \qquad \|u - u_h\|_{0,\Omega} \leqslant T' h^{l+1} |u|_{l+1,\Omega}$$

for another constant T' independent of h. In the case of a general polyhedral domain, the weaker estimate

$$(6.17) \qquad \|u - u_h\|_{0,\Omega} \leqslant T' h^l |u|_{l+1,\Omega}$$

follows from (6.13) and the analogue of the *Poincaré-Friedrichs inequality*

$$(6.18) \qquad \forall v_h \in V_h, \quad \|v_h\|_{0,\Omega} \leqslant c \|v_h\|_h,$$

where the constant $c = c(\Omega)$ depends only on Ω.

   In order to estimate the error $\lambda - \lambda_h$, we need first to check that hypothesis (6.7) holds with a constant $\alpha > 0$ independent of h.

   LEMMA 10. *Let there be given spaces X_h and M_h defined as in (3.11) and (3.15), which are associated with a regular family of triangulations. Assume that conditions (3.13) and (3.16) hold. Then hypothesis (6.7) holds with a constant $\alpha > 0$ independent of h.*

   Proof. For all $K \in T_h$ and all $\mu \in H^{-1/2}(\partial K)$, we define

$$(6.19) \qquad \||\mu\||_{\partial K} = \sup_{w \in H^l(K)} \frac{\int_{\partial K} \mu w \, d\gamma}{\||w\||_{1,K}}.$$

Then condition (3.16) means that there exists a constant $\hat{\alpha} > 0$ such that

$$(6.20) \qquad \forall \hat{\mu} \in \hat{S}, \quad \sup_{\hat{w} \in \hat{P}} \frac{\int_{\partial \hat{K}} \hat{\mu} \hat{w} \, d\gamma}{\||\hat{w}\||_{1,\hat{K}}} \geqslant \hat{\alpha} \||\hat{\mu}\||_{\partial \hat{K}}.$$

   Let $\mu_h \in M_h$. The first part of the proof consists in showing that, for each $K \in T_h$, there exists a function $v \in P_K$ such that

$$(6.21) \qquad \int_{\partial K} \mu_h v \, d\gamma \geqslant \hat{\alpha} \frac{\rho_{\hat{K}}}{h_{\hat{K}}} \frac{\rho_K}{h_K} \||\mu_h\||_{\partial K} \||v\||_{1,K}.$$

Consider the affine mapping $F_K : x \longrightarrow F_K(x) = B_K x + b_K$, where B_K is an invertible element of $L(R^n)$ and b_K is a vector in R^n, such that $K = F_K(\hat{K})$; we may write

$$(6.22) \qquad \int_{\partial K} \mu_h v \, d\gamma = \int_{\partial \hat{K}} \hat{\lambda} \hat{v} \, d\gamma,$$

where $\hat{v} = v_0 F_K \in \hat{P}$ and, by (3.13), $\hat{\lambda} \in \hat{S}$. Thus, we choose $\hat{v} \in \hat{P}$ such that

$$(6.23) \qquad \int_{\partial \hat{K}} \hat{\lambda} \hat{v} \, d\gamma \geqslant \hat{\alpha} \|\|\hat{\lambda}\|\|_{\partial \hat{K}} \|\|\hat{v}\|\|_{1, \hat{K}}.$$

Now, by using [5, Eq. (4.16) and Lemma 2], we obtain

$$\|\|v\|\|_{1,K}^2 = |v|_{1,K}^2 + h_K^{-2} \|v\|_{0,K}^2 \leqslant |\det(B_K)|(\|B_K^{-1}\|^2 |\hat{v}|_{1,\hat{K}}^2 + h_K^{-2} \|\hat{v}\|_{0,\hat{K}}^2)$$

$$\leqslant |\det(B_K)|(h_{\hat{K}}^2 \rho_K^{-2} |\hat{v}|_{1,\hat{K}}^2 + h_K^{-2} \|\hat{v}\|_{0,\hat{K}}^2)$$

so that

$$(6.24) \qquad \|\|v\|\|_{1,K} \leqslant h_{\hat{K}} \rho_K^{-1} |\det(B_K)|^{1/2} \|\|\hat{v}\|\|_{1,\hat{K}}.$$

On the other hand, it follows from (6.19) and (6.22) that

$$\|\|\mu_h\|\|_{\partial K} = \sup_{w \in H^l(K)} \frac{\int_{\partial \hat{K}} \hat{\lambda} \hat{w} \, d\gamma}{\|\|w\|\|_{1,K}}, \quad \hat{w} = w_0 F_K.$$

Since by [4, Eq. (4.15) and Lemma 2]

$$\|\|\hat{w}\|\|_{1,\hat{K}} \leqslant h_K \rho_{\hat{K}}^{-1} |\det(B_K)|^{-1/2} \|\|w\|\|_{1,K},$$

we get

$$(6.25) \qquad \|\|\mu_h\|\|_{\partial K} \leqslant h_K \rho_{\hat{K}}^{-1} |\det(B_K)|^{-1/2} \|\|\hat{\lambda}\|\|_{\partial \hat{K}}.$$

Hence inequality (6.21) follows from (6.22)–(6.25).

We shall now show that condition (6.7) follows from (6.21) with

$$\alpha = \frac{\hat{\alpha}}{\sigma} \frac{\rho_{\hat{K}}}{h_{\hat{K}}}.$$

Let $w \in H^l(K)$; by normalizing v in (6.21), there exists a function $v \in P_K$ such that

$$-\int_{\partial K} \mu_h v \, d\gamma \geqslant -\alpha \int_{\partial K} \mu_h w \, d\gamma, \quad \|\|v\|\|_{1,K} = \|\|w\|\|_{1,K}.$$

Therefore, given a function $w \in X$, there exists a function $v_h \in X_h$ such that

$$\frac{b(v_h, \mu_h)}{\|\|v_h\|\|_X} \geqslant \alpha \frac{b(w, \mu_h)}{\|\|w\|\|_X}.$$

Hence, we have

$$\sup_{v_h \in X_h} \frac{b(v_h, \mu_h)}{\|\|v_h\|\|_X} \geqslant \alpha \sup_{w \in X} \frac{b(w, \mu_h)}{\|\|w\|\|_X} = \alpha \|\|\mu_h\|\|_M. \quad \square$$

We are now able to prove

THEOREM 5. *Assume the hypotheses of Theorem 4. Assume, in addition, that conditions* (3.13) *and* (3.16) *hold. Then there exists a constant* $T > 0$ *independent of h such that*

(6.26) $$\||\lambda - \lambda_h\||_M \leqslant Th^l|u|_{l+1,\Omega}.$$

Proof. Using Lemma 9 with $\varphi = u$ and $\psi = \lambda$ we obtain

$$\inf_{\mu_h \in M_h} \sup_{v \in X} \frac{b(v, \lambda - \mu_h)}{\|v\|_h} \leqslant Th^l|u|_{l+1,\Omega}.$$

Since $\|v\|_h \leqslant \||v\||_X$ and by the definition (6.4), we get

(6.27) $$\inf_{\mu_h \in M_h} \||\lambda - \mu_h\||_M \leqslant Th^l|u|_{l+1,\Omega}.$$

Then inequality (6.26) follows from Lemma 10 and the inequalities (6.8), (6.13) and (6.27). □

Remark 6. In fact, one can derive a more "suggestive" result (cf. [18]). Define the following norm over the space $\Pi_{K \in T_h} L^2(\partial K)$.

(6.28) $$\|\mu\|_h = \left(\sum_{K \in T_h} h_K \|\mu\|_{0,\partial K}^2 \right)^{1/2}.$$

Then one can prove (cf. [18]) that there exists a constant $c = c(\Omega) > 0$ such that

(6.29) $$\forall \mu_h \in M_h, \quad \|\mu_h\|_h \leqslant C\||\mu_h\||_M,$$

so that we get as a consequence of the inequalities (6.26) and (6.29)

(6.30) $$\|\lambda - \lambda_h\|_h \leqslant Th^l|u|_{l+1,\Omega}. \quad □$$

Let us go back to the Examples 1, 2, 3. Assume that $u \in H^{l+1}(\Omega)$ for some integer l with $1 \leqslant l \leqslant k$ and that (T_h) is a regular family of triangulations. It follows from Theorems 4 and 5 that in each example, we have

$$\|u - u_h\|_h + \||\lambda - \lambda_h\||_M \leqslant Th^l|u|_{l+1,\Omega}.$$

7. Remarks on Some Nonconforming Methods. Let us go back to the general formulation of primal hybrid methods. It can be of interest to use numerical quadrature for evaluating the various surface integrals $\int_K \mu v \, d\gamma$ which appear in the bilinear form $b(v, \mu)$. Then we obtain a new bilinear form $b_h(v, \mu)$ and we replace problem (Q_h) by the following one, called *problem* (Q_h^*): Find a pair $(u_h^*, \lambda_h^*) \in X_h \times M_h$ such that

(7.1) $$\forall v_h \in X_h, \quad a(u_h^*, v_h) + b_h(v_h, \lambda_h^*) = \int_\Omega fv_h \, dx,$$

(7.2) $$\forall \mu_h \in M_h, \quad b_h(u_h^*, \mu_h) = 0.$$

With the bilinear form $b_h(v, \mu)$, we associate the subspace V_h^* of the space X_h

(7.3) $$V_h^* = \{v_h \in X_h; \ \forall \mu_h \in M_h, \ b_h(v_h, \mu_h) = 0\}$$

and we define *problem* (P$_h^*$): Find $u_h^* \in V_h^*$ such that

(7.4) $$\forall v_h \in V_h^*, \ a(u_h^*, v_h) = \int_\Omega f v_h \ dx.$$

Here again, if $(u_h^*, \lambda_h^*) \in X_h \times M_h$ is a solution of problem (Q$_h^*$), $u_h^* \in V_h^*$ is a solution of problem (P$_h^*$). Now, the degrees of freedom of a function $v_h \in V_h^*$ can be easier to determine than those of a function $v_h \in V_h$ so that problem (P$_h^*$) can be simpler to solve than problem (P$_h$).

This is indeed the case in some important examples that we shall discuss briefly. Again, we refer to [18] for general results concerning the existence, uniqueness and approximation properties of the solutions of problems (Q$_h^*$) and (P$_h^*$).

Example 4. Consider again the situation of Example 2 where $\hat{K}$ is a triangle, k is an even integer and the pair $(\hat{P}, \hat{S}_{k-1})$ is chosen as in Lemma 6. But here, for each side K' of T_h, we use the k-points Gauss-Legendre quadrature formula to compute the integral $\int_{K'} \mu_h v_h \ d\gamma$, $v_h \in X_h$, $\mu_h \in M_h$. In fact, one can prove that the corresponding problem (Q$_h^*$) has a unique solution (u_h^*, λ_h^*) and that, for $u \in H^k(\Omega)$, we have the error estimate

(7.5) $$\|u - u_h^*\|_h + |||\lambda - \lambda_h^*|||_M \leqslant Th^{k-1} |u|_{k, \Omega}.$$

Note that the order of accuracy of the hybrid method has been decreased by one. On the other hand, the space V_h^* can now be characterized as in Example 1: a function $v_h \in X_h$ belongs to V_h^* iff it satisfies conditions (4.9) and (4.10). Moreover, the space V_h^* is associated with a nonconforming reference finite element $(\hat{K}, \hat{\Sigma}, \hat{P})$. Assume for simplicity that $k = 2$. Then:

  (i) $\hat{\Sigma} = \{b_i\}_{i=1}^7$ where the points b_i, $1 \leqslant i \leqslant 6$, are the Gauss-Legendre quadrature points and where b_7 may be chosen as the centroid of $\hat{K}$;

  (ii) $\hat{P}$ is the space of polynomials spanned by P_2 and the function $v_0 = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)$.
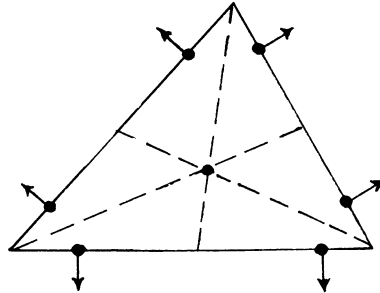

FIGURE 2

We recognize here a nonconforming element introduced by Irons and Razzaque [8, p. 579].

Example 5. We next consider the situation of Example 3: $\hat{K}$ is the unit square, k is any integer $\geqslant 1$ and the pair $(\hat{P}, \hat{S}_{k-1})$ is chosen as in Lemma 8. Again we compute each integral $\int_{K'} \mu_h v_h \ d\gamma$ by using the k-points Gauss-Legendre quadrature

formula. Then the problem (Q_h^*) has a unique solution and, for $u \in H^k(\Omega)$, we have
the estimate (7.5). Note that the convergence of this hybrid method is ensured only
for $k \geq 2$.

For instance, when $k = 2$ and $\hat{Q}$ is the space of polynomials spanned by $1, \xi, \eta$,
$\xi^2, \xi\eta, \eta^2, \xi^2\eta, \xi\eta^2$, the space V_h^* is associated with a nonconforming reference ele-
ment $(\hat{K}, \hat{\Sigma}, \hat{P})$ such that

(i) $\hat{\Sigma} = \{b_i\}_{i=1}^9$ where the points b_i, $1 \leq i \leq 8$, are the Gauss-Legendre quad-
rature points and where b_9 is the centroid of $\hat{K}$;

(ii) $\hat{P}$ is the space of polynomials spanned by $\hat{Q}$ and the function $v_0(\xi, \eta) = $
$[\xi(1 - \xi) - \eta(1 - \eta)](2\xi - 1)(2\eta - 1)$.

Here again, we obtain a nonconforming element introduced by Irons and
Razzaque [8, p. 579].

Example 6 (Wilson's rectangular element [19]). Here $\hat{K}$ is the unit square; we
set $\hat{P} = P_2$ and $\hat{S} = \hat{S}_1$. Unfortunately, by Lemma 7, the compatibility condition
(3.16) is not satisfied so that the problem (Q_h) is not well posed. However, by Lem-
ma 2, problem (P_h) has a unique solution $u_h \in V_h$ and, by using Theorem 4 with k
$= 1$ and $m = 0$, we get the estimate

$$(7.6) \qquad \|u - u_h\|_h \leq Th|u|_{2,\Omega}.$$

Now, instead of computing exactly each integral $\int_{K'} \mu_h v_h \, d\gamma$, we use the trape-
zoidal rule, the quadrature nodes being the endpoints of K'. Then, one can prove
that problem (P_h^*) has a unique solution $u_h^* \in V_h^*$ and that, for $u \in H^2(\Omega)$, we have

$$(7.7) \qquad \|u - u_h^*\|_h \leq Th|u|_{2,\Omega}.$$

Furthermore, the space V_h^* may be characterized as the space of all functions v_h
such that

(i) for each rectangle $K \in T_h$, $v_{h|K} \in P_2$;

(ii) v_h is continuous at the vertices of T_h contained in Ω;

(iii) v_h vanishes at the vertices located on Γ.

Hence the space V_h^* is associated with Wilson's rectangular element as it is described
in [14]. For direct proofs of convergence of this nonconforming method, we refer
to [14], [9].

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1. F. BREZZI, "On the existence, uniqueness and approximation of saddle-point problems
arising from Lagrangian multipliers," *Rev. Française Automat. Informat. Recherche Opérationelle
Sér. Anal. Numér.*, v. 8, 1974, no. R-2, pp. 129–151. MR 51 #1540.

2. F. BREZZI, "Sur la méthode des éléments finis hybrides pour le problème biharmo-
nique," *Numer. Math.*, v. 24, 1975, pp. 103–151.

3. F. BREZZI & L. D. MARINI, "On the numerical solution of plate bending problems by
hybrid methods," *Rev. Française Automat. Informat. Recherche Opérationelle Sér. Anal. Numér.*,
v. 9, 1975, no. R-3, pp. 5–50.

4. P. G. CIARLET, *Numerical Analysis of the Finite Element Method*, Séminaire de Mathé-
matiques Supérieures, Univ. de Montréal Press, Montréal, 1975.

5. P. G. CIARLET & P. A. RAVIART, "General Lagrange and Hermite interpolation in R^n with applications to finite element methods," *Arch. Rational Mech. Anal.,* v. 46, 1972, pp. 177–199.   MR 49 #1730.

6. M. CROUZEIX & P. A. RAVIART, "Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I," *Rev. Française Automat. Informat. Recherch Opérationelle Sér. Anal. Numér.,* v. 7, 1973, no. R-3, pp. 33–75.   MR 49 #8401.

7. B. FRAEIJS DE VEUBEKE, "Variational principles and the patch test," *Internat. J. Numer. Methods Engrg.,* v. 8, 1974, pp. 783–801.   MR 51 #12099.

8. B. M. IRONS & A. RAZZAQUE, "Experience with the patch test for convergence of finite elements," *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations,* Part II (A. K. Aziz, Editor), Academic Press, New York, 1972, pp. 557–587.   MR 49 #11824.

9. P. LESAINT, "On the convergence of Wilson's nonconforming element for solving the elastic problems," *Comput. Methods Appl. Mech. Engrg.,* v. 7, 1976, pp. 1–16.

10. T. H. H. PIAN, "Formulations of finite element methods for solid continua," *Recent Advances in Matrix Methods of Structural Analysis and Design* (R. H. Gallagher, Y. Yamada and J. T. Oden, Editors), The University of Alabama Press, 1971, pp. 49–83.

11. T. H. H. PIAN, "Finite element formulation by variational principles with relaxed continuity requirements," *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations,* Part II (A. K. Aziz, Editor), Academic Press, New York, 1972, pp. 671–687.   MR 49 #11824.

12. T. H. H. PIAN & P. TONG, "Basis of finite element methods for solid continua," *Internat. J. Numer. Methods. Engrg.,* v. 1, 1969, pp. 3–28.

13. P. A. RAVIART, "Hybrid finite element methods for solving 2nd order elliptic equations," *Conf. on Numerical Analysis,* Royal Irish Academy, Dublin, 1974.

14. G. STRANG, "Variational crimes in the finite element methods," *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations,* Part II (A. K. Aziz, Editor), Academic Press, New York, 1972, pp. 689–710.   MR 49 #11824.

15. G. STRANG & G. J. FIX, *An Analysis of the Finite Element Method,* Prentice-Hall, Englewood Cliffs, N. J., 1973.

16. J. M. THOMAS, "Méthode des éléments finis hybrides duaux pour les problèmes elliptiques du 2nd ordre," *Rev. Française Automat. Informat. Recherche Opérationelle Sér. Anal. Numér.,* v. 10, 1976, pp. 51–79.

17. J. M. THOMAS, *Méthode des Eléments Finis Equilibre pour les Problèmes Elliptiques du 2nd Ordre,* Journées des Eléments Finis, May 12–14 1975, Rennes, France.

18. J. M. THOMAS, Thesis, Univ. P. & M. Curie, Paris, 1977.

19. E. L. WILSON, R. L. TAYLOR, W. P. DOHERTY & J. GHABOUSSI, "Incompatible displacement models," *Sympos. on Numerical Methods in Structural Engineering,* O. N. R., University of Illinois, 1971.

20. O. C. ZIENKIEWICZ, *The Finite Element Method in Engineering Science,* 2nd rev. ed., McGraw-Hill, London, 1971.   MR 47 #4518.